ECS 20 – Fall 2021 – P. Rogaway Counting

Today: Let's count!

First, the "theory"

2^n = Number of subsets of n items
= number of n-bit binary strings
= number of ways to paint n items with d different colors.

d^n = Number of length-n strings over an alphabet of d character
= number of ways to paint n items with d different colors.

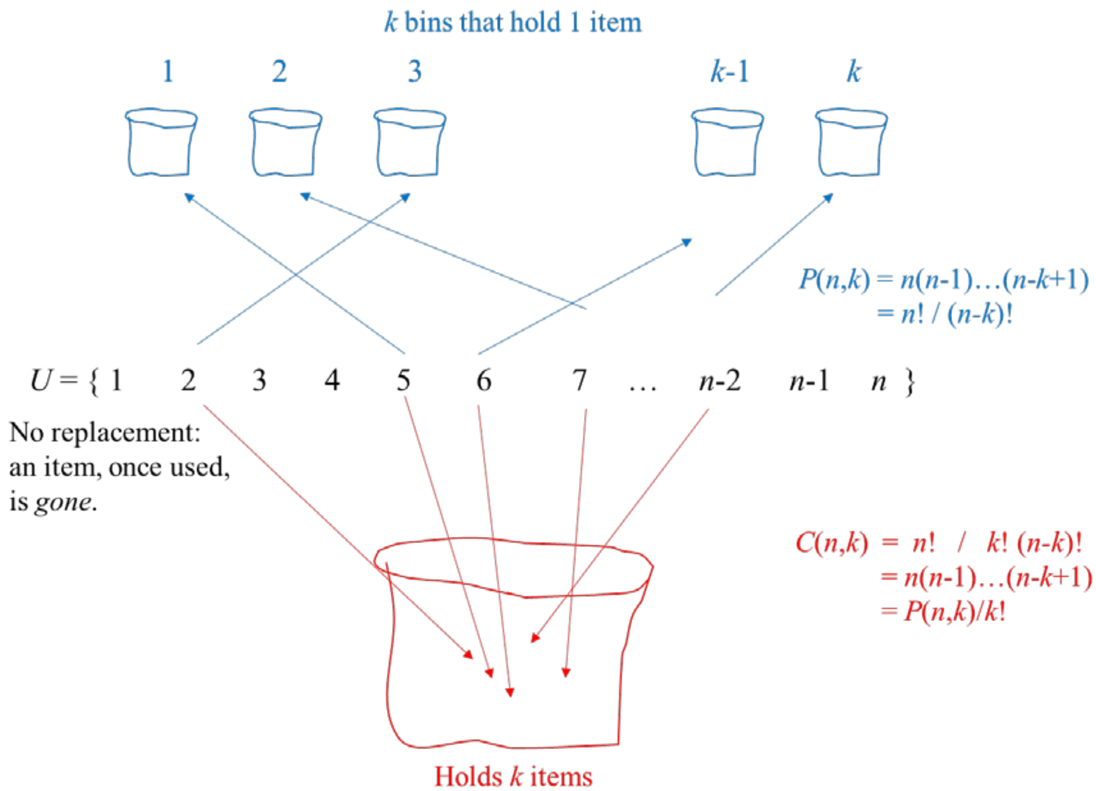
$n!$ = Number of ways to arrange n different items
= Number of ways to order {1,2,…,n

$P(n, k)$ = The number of ways to arrange k items drawn, without replacement, from a universe of n items
= Number of ways to fill k bins, one item per bin, from a universe $\{1,...,n\}$
= $n(n-1) \ldots (n-k+1)$
= $n!/(n-k)!$
No replacement; an item, once used, is gone.

$C(n, k)$ = Number of ways to fill a bin with k items from a universe $\{1,...,n\}$
= number of k-element subsets from a set of n different items
= $n! \;/\; k!(n-k)!$
= $P(n, k)/k!$
No replacement; an item, once used, is gone.
Supported by Google's search-line calculator as in "100 choose 50"

Alternate notation: $\binom{n}{k}$

$C(n, 2)$ = Number of 2-element subsets from an n-element set
= number of k-element subsets from a set of n different items
= $n(n\text{-}1)/2$



product rule = if event A can occur in a ways and, independent of this,
event B can occur in b ways
then the number of combinations of ways
for A and B to occur is ab.

- Really just a statement that $|A \times B| = |A|\ |B|$ for finite A, B.

sum rule = if event A can occur in a ways and
event B can occur in b ways,
but both events cannot occur together,
then the number of ways for A **or** B to occur is a+b.

- Really just a statement that $|A \cup B| = |A|+|B|$ for disjoint A, B.

inclusion/exclusion counting:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

And generalizations, like

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Reminder: $\log(n!) \approx n \log n$

Example counting exercises

Please calculate values explicitly to the point of getting out numbers – I like to see actual numbers.

1. How many ways can a blue, white, and red ball be put into 10 different bins? Assume no bin can contain two balls.

   Answer: 10*9*8 = $P(10,3)$ = 720

2. License plates in Nebraska are 3 distinct letters (A-Z, but not O), followed by 3 distinct decimal digits. How many possible license plates are there?

   Answer: 25*24*23*10*9*8 = $P(25,3)$ $P(10,3)$ = 9,936,000

3. How many different ways a salesman travel among n cities, where he starts in city 1 and visits each other city once and only once before returning to city 1.

   Answer: $(n - 1)!$

4. How many ways can you select a president, vice president, and treasurer in a club of 30 people?

   Answer: $P(30,3)$ = 24,360

5. How many way can you form Male-Female dance partners if there are 12 women and 8 men. Assume each man is partnered with some woman (4 women go un-partnered).

   Answer: $P(12,8)$ = 19,958,400

6. How many ways you position 7 people in a circle?

Answer: 6! = 720

7. A man, a woman, a boy, a girl, a dog, and a cat are walking single-file down the road.

a. How many ways can this happen?
Answer: 6! = 720

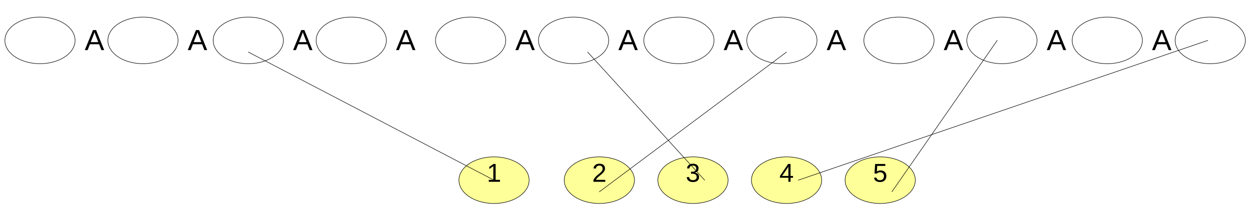
b. How many ways if the dog comes first?
Answer: 5! = 120

c. How many ways if the dog immediately follows the boy?
Answer: 5! = 120

d. How many ways if the dog (and only the dog) is immediately between the man and the boy.
Answer: 2*4! = 48 (form a man-dog-boy or a boy-dog-man combo)
(so walking down the street is a woman, a girl, a cat, and a man-dog-boy (4!)
or, walking down the street is a woman, a girl, a cat, and a boy-dog-man (4!)

8. In how many ways can 10 adults and 5 children be positioned in a line so that no two children are next to each other? (they fight)

Answer: $10!*P(11,5) = 10!\ 11!\ /\ 6! = 201{,}180{,}672{,}000 \approx 10^{11.3}$



9. How many arrangements are there of the letters a..z such that there are exactly 10 letters between the "A" and the "Z"?

Answer: $15!*P(24,10)*2 = 24!*30 \approx 1.86*10^{25}$

(reasoning: after selecting the AxxxxxxxxxxZ block, treat it as atomic and rearrange it with the 14 remaining letters in any of 15! ways. Double to account for both the AxxxxxxxxxxZ and ZxxxxxxxxxxA possibilities.)

10. You take a group of four people to a Chinese restaurant that has 100 different dishes. All food will be shared among the four of you. How many ways can you order 4 different dishes?

Answer: $C(100,4)$ = 100*99*98*97 / (4*3*2*1) = 3,921,225

11. You toss a coin 8 times. How many ways can it land with 5 heads total?

Answer: $C(8,5) = 56$
(Note this is $C(8,3)$. In general, $C(n, k) = C(n, n\text{-}k)$.)

12. How many 6-element subsets are there of the letters, A ... Z ?

$C(26,6) = 230,230$

How many 2-element subsets are there of the letters A ... Z ?
$C(26,2) = 26*25/2 = 325$.
In general, $C(n,2) = n(n-1)/2$

How many subsets are there of the letters A … Z ?

$2^{26} = 67,108,864$

13. An urn contains 15 red, distinctly numbered, balls, and 10 white, distinctly numbered balls.
5 balls are removed.

(A) How many different samples are possible?

Answer: $C(25,5) = 53,130$

(B) How many samples contain only red balls?

Answer: $C(15,5) = 3003$.

(B') So what is the **probability** that a random sample will contain only red balls?

Answer: 3003 / 53,130 ≈ 0.05652 (05.652 %) (a little more than about 1 in 18)

(C) How many samples contains 3 red balls and 2 white balls?

Answer: $C(15,3) * C(10,2) = 20,475$

(C') So what's the chance that a random sample will contain 3 red balls and one white ball?

Answer: 20,475 / 53,130 ≈ 0.3854 (38.54%)

14. How many numbers are there between 1 and 1000 have are not divisible by 3, 5, or 7

A_3 = numbers in [1..1000] that are divisible by 3. $|A_3|=333$
A_5 = numbers in [1..1000] that are divisible by 5. $|A_5|=200$
A_7 = numbers in [1..1000] that are divisible by 7. $|A_7|=\lfloor 1000/7 \rfloor=142$
$A_{3,5}$ = numbers in [1..1000] that are divisible by 3 & 5. $|A_{3,5}| = \lfloor 1000/15 \rfloor = 66$
$A_{5,7}$ = numbers in [1..1000] that are divisible by 5 & 7. $|A_{3,5}| = \lfloor 1000/35 \rfloor = 28$
$A_{3,7}$ = numbers in [1..1000] that are divisible by 3 & 7. $|A_{3,5}| = \lfloor 1000/21 \rfloor = 47$
$A_{3,5,7}$ = nums in [1..1000] that are divisible by 3&5& 7. $|A_{3,5,7}| = \lfloor 1000/3*5*7 \rfloor = 9$
So answer, by inclusion, exclusion, is 1000–333–200–142 +66+28+47–9 = 457

15. Poker. Deck of 52 cards, these having 13 “values” and 4 suits. 5 cards are dealt. We are interested in the probability of being dealt certain kinds of hands.

royal flush = 10JQKA of one suit.
straight flush = five consecutive cards: 2345, , ..., , 10JQKA in any suit.
four of a kind = four cards of one value (e.g., all four 9's)
full house = 3 cards of one value, 2 cards of another value. (Eg, 3xA, 2x4).
flush = five cards of a single suit
three of a kind = 3 cards of one value, a fourth card of a different value, and a fifth card of a third value

two pairs = two cards of one value, two more cards of a second value,
and the remaining card of a third value
one pair = two cards of one value, but not classified above

a) How many poker hands are there?

Answer: $C(52,5)$=2,598,960

b) How many poker hands are full houses?

Answer: A full house can be **partially** identified by a pair, like (J,8), where the first component of the pair is what you have **three** of, the second component is what you have **two** of. So there are $P(13,2)$=13*12 such pairs. For each there are $C(4,3)$=4 ways to choose the first component, and $C(4,2)$=6 ways to choose the second component. So all together there are **13*12*4*6=3,744 possible full houses.**

c) What's the probability of being dealt a full house?

3,744/2,598,960 ≈ 0 .001441 ≈ 0.14%

$$P[\text{FullHouse}] \approx .001441$$

The **probability** of an event is a real number between 0 and 1 (inclusive). If asked what's the probability of something, don't answer with a "percent", and don't answer with something outside of [0,1]. When we give something in "percent's", we are giving a probability multiplied by 100.

d) How many poker hands are two pairs?

Answer: We can partially identify two pairs as in {J, 8}. Note that now the pair is now **unordered**. There are $C(13,2)$ such sets. For each there are $C(4,2)$ ways to choose the larger card and $C(4,2)$ ways to choose the smaller card. There are now 52 - 8 =44 remaining cards one can choose as the fifth card (to avoid a full house, there are 8 "forbidden" cards). So the total is

$$C(13,2)*C(4,2)*C(4,2)*44 = 123{,}552.$$

e) What is the probability of being dealt two pairs?

$C(13,2)*C(4,2)*C(4,2)*44 / C(52,5) =$ 123,552/2,598,960
$\approx 0.047539 \quad \approx 4.75\%$

P[TwoPairs] 0.047539